

Unified derivation of the limit shape for a “conservative” ensemble of random partitions

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Abstract

We derive the limit shape of Young diagrams, associated with growing integer partitions, for a wide class of “conservative” multiplicative probability measures underpinned by the generating functions of the form $F(z) = \prod_{\ell=1}^{\infty} f(z^{\ell})$ ($0 < z < 1$). Under mild technical assumptions on the function $H(s) = \ln f(s)$, we show that the limit shape ω^* is given by the equation $y = \kappa^{-1} H(e^{-\kappa x})$, where $\kappa^2 = \int_0^1 s^{-1} H(s) ds$. The conservative class covered by this result includes (but is not limited to) analogues of the three meta-types of decomposable combinatorial structures — multisets, selections and assemblies. Our method is based on the usual randomization and conditioning device; to this end, a suitable local limit theorem is proved.

Key words and phrases: Integer partition; Young diagram; limit shape; local limit theorem

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1. Introduction

An *integer partition* is a decomposition of a given natural number into an unordered sum of integers; for example, $12 = 4 + 2 + 2 + 2 + 1 + 1$. More formally, a collection of integers $\lambda = \{\lambda_i \in \mathbb{N} : \lambda_1 \geq \lambda_2 \geq \dots > 0\}$ is a partition of $n \in \mathbb{N}$ if $n = \lambda_1 + \lambda_2 + \dots$, which is sometimes denoted as $\lambda \vdash n$. The terms $\lambda_i \in \lambda$ are called *parts* of the partition λ . An alternative notation $\lambda = (1^{\nu_1} 2^{\nu_2} \dots)$ specifies the *multiplicities* (counts) of the parts involved, $\nu_{\ell} := \#\{\lambda_i \in \lambda : \lambda_i = \ell\} = \sum_{\lambda_i \in \lambda} \mathbf{1}_{\{\ell\}}(\lambda_i)$ ($\ell \in \mathbb{N}$). We denote by Π_n the (finite) set of all partitions $\lambda \vdash n \in \mathbb{N}$.

Partitions λ are succinctly visualized by their *Young diagrams* Υ_{λ} formed by (left- and bottom-aligned) row blocks with $\lambda_1, \lambda_2, \dots$ unit square cells, respectively, representing the parts (see Fig. 1a). If $\lambda \in \Pi_n$ then the area of the Young diagram Υ_{λ} equals n . The upper boundary of Υ_{λ} is a piecewise constant function $Y_{\lambda} : \mathbb{R}_+ \rightarrow \mathbb{Z}_+$ given by (see Fig. 1b)

$$Y_{\lambda}(x) := \sum_{\ell > x} \nu_{\ell}, \quad \lambda \leftrightarrow \{\nu_{\ell}, \ell \in \mathbb{N}\}. \quad (1.1)$$

In particular, $Y_{\lambda}(0) = \sum_{\ell=1}^{\infty} \nu_{\ell} = \#\{\lambda_i \in \lambda\}$ gives the “length” of partition λ (i.e., the total number of parts involved).

If the space Π_n is endowed with a probability measure P_n (for example, the uniform measure where all $\lambda \in \Pi_n$ are equiprobable) then one can talk about *random* partitions. The *limit shape*, with respect to a probability measure P_n on Π_n as $n \rightarrow \infty$, is understood as (the graph of) a function $y = \omega^*(x)$ such that, for each $\delta > 0$ and any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} P_n \left\{ \lambda \in \Pi_n : \sup_{x \geq \delta} |\tilde{Y}_{\lambda}^n(x) - \omega^*(x)| \leq \varepsilon \right\} = 1, \quad (1.2)$$

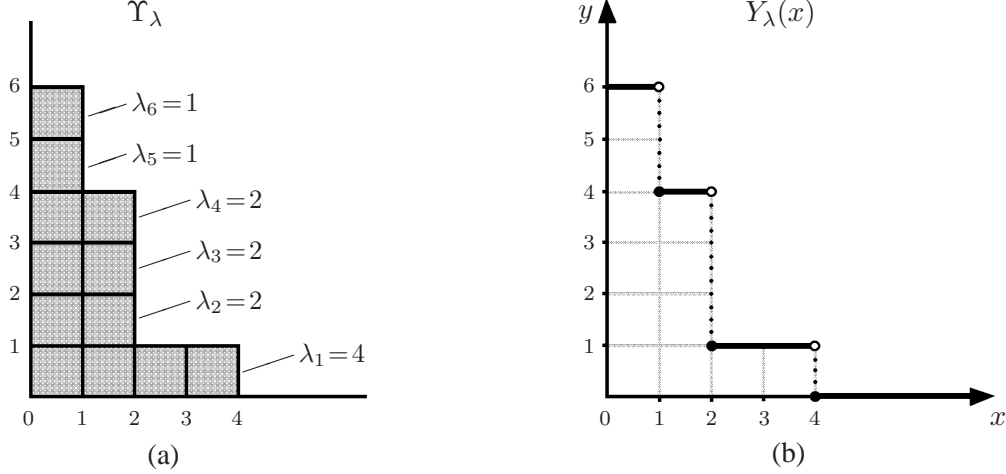


Figure 1: The Young diagram Υ_λ (a) and the graph of the corresponding (upper boundary) function $Y_\lambda(x) = \sum_{\ell > x} \nu_\ell$ (b) for a partition $\lambda = (4, 2, 2, 2, 1, 1) \equiv (1^2 2^3 4^1)$ of $n = 12$, with the part multiplicities $\nu_1 = 2$, $\nu_2 = 3$ and $\nu_4 = 1$.

where $\tilde{Y}_\lambda^n(x) = A_n^{-1} Y_\lambda(x B_n)$ for suitable scaling coefficients A_n, B_n . It is natural to require that $A_n B_n = n$, which would render the area of the scaled Young diagram $\tilde{\Upsilon}_\lambda$ to be normalized to unity; the most frequent choice is specified as $A_n = B_n = n^{1/2}$.

Of course, the limit shape and its very existence depends on the choice of a probability law P_n on Π_n . With respect to the uniform (equiprobable) distribution on Π_n , the limit shape ω^* , first identified (on a physical level of rigor) by Temperley [19] in relation to the equilibrium shape of a growing crystal, and later on obtained rigorously by Vershik (see [25, p.30]), is determined by the equation (see Fig. 2a)

$$e^{-\pi x/\sqrt{6}} + e^{-\pi y/\sqrt{6}} = 1. \quad (1.3)$$

Full proof of this result, in its modern form, was published by Vershik in paper [22], which also covered a few other partition ensembles of the so-called *multiplicative type*, including the uniform distribution on the subset $\check{\Pi}_n \subset \Pi_n$ of partitions with distinct parts (i.e., restricted by the condition $\nu_\ell \leq 1$), where the limit shape is given by the equation (see Fig. 2b)

$$e^{\pi y/\sqrt{12}} = 1 + e^{-\pi x/\sqrt{12}}. \quad (1.4)$$

For a general definition and details of multiplicative probability measures on partitions, see [12, 22, 23]; in short, such measures are underpinned by the generating functions of the form

$$F(z) = \prod_{\ell=1}^{\infty} f_\ell(z^\ell), \quad |z| < 1, \quad (1.5)$$

where the factors $f_\ell(\cdot)$ determine a certain weighting of the counts ν_ℓ ($\ell \in \mathbb{N}$) in a random partition $\lambda \in \Pi_n$. For example, the uniform distribution on the partition spaces Π_n and $\check{\Pi}_n$ corresponds to the choice $f_\ell(s) \equiv f(s) = (1-s)^{-1}$ or $f_\ell(s) \equiv f(s) = 1+s$, respectively.

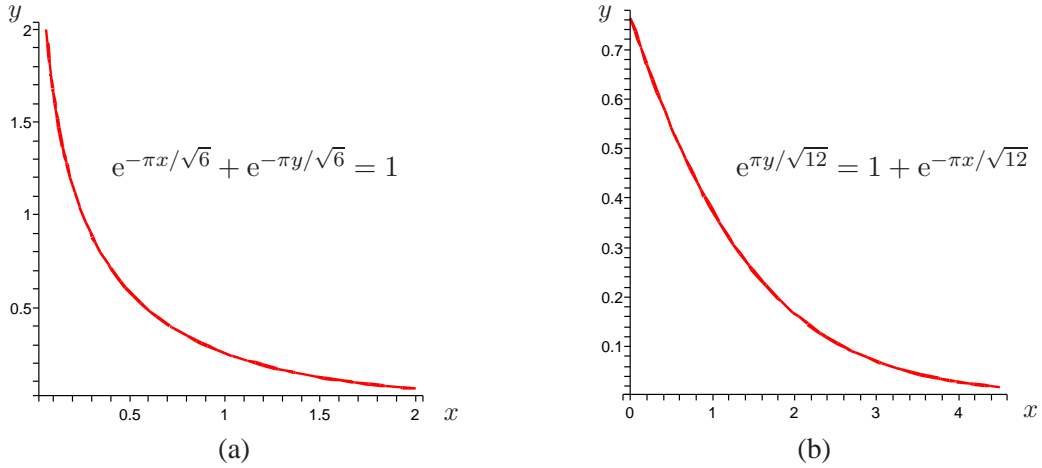


Figure 2: The limit shape ω^* for two classical ensembles of uniform (equiprobable) random partitions: (a) unrestricted partitions (Π_n), $f(s) = (1 - s)^{-1}$; (b) partitions with distinct parts ($\tilde{\Pi}_n$), $f(s) = 1 + s$.

Building on the ideas coined by Vershik, the limit shape problem was advanced in various directions (see, e.g., [4, 7, 8, 10, 11, 12, 17, 18, 23, 26] and further references therein). In a separate but related development, Logan and Shepp [16] and Vershik and Kerov [24, 25] found the limit shape for a different (non-multiplicative) ensemble of partitions endowed with the *Plancherel measure* emerging in relation with representation theory of the symmetric group.

Returning to the multiplicative class of probability measures on partitions, note that most of the aforementioned papers on the limit shape problem have focused on the particular case $f_\ell(s) = f(s)^{b_\ell}$ for some classes of sequences $b_\ell > 0$ but with a more limited choice of the basic function $f(s)$, usually borrowed from the standard equiprobable cases mentioned above. A recent paper by Yakubovich [28] offers a more general treatment by considering a wider class of functions $f(s)$; a typical condition imposed there on such a function is its analyticity up to an isolated singularity point $s_1 \geq 1$, which *must be a pole* if $s_1 = 1$. Many examples such as $f(s) = (1 - s)^{-r}$ with a real (non-integer) $r > 0$ can still be fitted in this condition by representing¹⁾ $f(s) = \tilde{f}(s)^r$, where $\tilde{f}(s) := (1 - s)^{-1}$ has a required (simple) pole at $s_1 = 1$; however, there are examples with a genuine non-pole singularity that do possess a limit shape (see Example 2.4 in Sections 2.3 and 7.2).

In the present paper, we restrict ourselves to the “conservative” case of multiplicative measures, specified by the simplest choice $b_\ell \equiv 1$ but with a fairly general variety of permissible functions $f(s)$. In particular, measures covered by our method include (but are not limited to) direct analogues of the three classical meta-types of decomposable combinatorial structures — multisets, selections and assemblies [1, 2] (see examples in Section 2.3 below).

Let us state our result more precisely. Assume that the measures P_n on Π_n are determined by (1.5) with $f_\ell(s) \equiv f(s) = \sum_{k=0}^{\infty} c_k s^k$ ($|s| < 1$), such that $c_0 = 1$ and all $c_k \geq 0$. More

¹⁾Incidentally, this remark shows that it is more natural to set conditions on the function $H(s) := \ln f(s)$ rather than on $f(s)$ itself.

precisely, for each partition $\lambda \in \Pi_n$ encoded via the counts $\{\nu_\ell, \ell \in \mathbb{N}\}$, let us set

$$P_n(\lambda) := \frac{c(\lambda)}{\mathfrak{C}_n}, \quad c(\lambda) := \prod_{\ell=1}^{\infty} c_{\nu_\ell}, \quad \mathfrak{C}_n := \sum_{\lambda \in \Pi_n} c(\lambda). \quad (1.6)$$

Denote $H(s) := \ln f(s)$, $\kappa := \int_0^1 s^{-1} H(s) \, ds$, and consider the function

$$\omega^*(x) := \kappa^{-1} H(e^{-\kappa x}), \quad x \geq 0. \quad (1.7)$$

A loose formulation of our main result is as follows.

Theorem 1.1. *Under some mild technical conditions on the function $H(s)$ (see more details in Section 2.1), for each $\delta > 0$ and any $\varepsilon > 0$*

$$\lim_{n \rightarrow \infty} P_n \left\{ \lambda \in \Pi_n : \sup_{x \geq \delta} \left| \tilde{Y}_\lambda^n(x) - \omega^*(x) \right| \leq \varepsilon \right\} = 1, \quad (1.8)$$

where $\tilde{Y}_\lambda^n(x) := n^{-1/2} Y_\lambda(x n^{1/2})$ and the limit shape function ω^* is defined in (1.7).

Remark 1.1. The restriction $x \geq \delta > 0$ in (1.8) takes into account the possibility $\omega^*(0) = \infty$ (cf. (1.3), (1.4)). Whenever $\omega^*(0) < \infty$, the supremum in (1.8) can be extended to all $x > 0$.

Like in [7, 9, 22, 23, 28], our proof employs the elegant probabilistic approach in the theory of decomposable combinatorial structures based on randomization and conditioning, first applied in the context of random partitions by Fristedt [9] (see a recent monograph [1] and an earlier review [2] for a general discussion of the method and many examples). The idea is to introduce a suitable measure Q_z on the union space $\Pi = \cup_n \Pi_n$ (depending on an auxiliary “free” parameter $z \in (0, 1)$), such that a given measure P_n on Π_n is recovered as the conditional distribution $P_n(\cdot) = Q_z(\cdot | \Pi_n)$. The great advantage of multiplicativity property (1.5) is that Q_z can be constructed as a product measure, resulting in *independent* random counts ν_ℓ . Clearly, such a device calls for the asymptotics of the probability $Q_z(\Pi_n)$, which is supplied by proving a suitable local limit theorem. Let us also point out that the parameter z is calibrated from the asymptotic equation $E_z(N_\lambda) = n(1 + o(1))$, where $N_\lambda := \lambda_1 + \lambda_2 + \dots = \sum_{\ell=1}^{\infty} \ell \nu_\ell$ is the “size” of the random partition $\lambda \in \Pi$ (so that, e.g., $\Pi_n = \{\lambda \in \Pi : N_\lambda = n\}$). This is sufficient to ensure the (uniform) convergence of the *expectation* $E_z[\tilde{Y}_\lambda^n(\cdot)]$ to $\omega^*(\cdot)$, but in order to extend this to the random paths $\tilde{Y}_\lambda^n(\cdot)$, our methods requires an improved estimate of the approximation error $E_z(N_\lambda) - n$ of at least the order of $o(n^{1/2})$.

Layout. The paper is organized as follows. In Section 2, we define the families of measures Q_z and P_n on the corresponding spaces of partitions. In Section 3, a suitable value of the parameter $z \in (0, 1)$ is chosen (Theorem 3.1), which implies convergence of “expected” (scaled) Young diagrams to the limit curve $y = \omega^*(x)$ (Theorem 3.3). Refined first-order moment asymptotics are obtained in Section 4 (Theorem 4.1), while higher-order moment sums are analyzed in Section 5. Section 6 is devoted to the proof of a local central limit theorem (Theorem 6.1). Finally, the limit shape result, with respect to both Q_z and P_n , is proved in Section 7.1 (Theorems 7.1 and 7.2, respectively), illustrated by some examples in Section 7.2.

Some notations. We denote $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{Z}_+ := \{k \in \mathbb{Z} : k \geq 0\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. The notation $x_n \asymp y_n$ as $n \rightarrow \infty$ means that $0 < \liminf x_n/y_n \leq \limsup x_n/y_n < \infty$. We also use the standard notation $x_n \sim y_n$ for $x_n/y_n \rightarrow 1$.

2. Probability measures on spaces of partitions

2.1. Global measure Q_z and conditional measure P_n

Let $\Phi := (\mathbb{Z}_+)^{\mathbb{N}}$ be the space of functions on \mathbb{N} (i.e., sequences) with nonnegative integer values, and consider the subspace of functions with *finite support* $\Phi_0 := \{\nu \in \Phi : \#(\text{supp } \nu) < \infty\}$, where $\text{supp } \nu := \{\ell \in \mathbb{N} : \nu_\ell > 0\}$. The space Φ_0 is in one-to-one correspondence with the set $\Pi = \bigcup_{n \in \mathbb{Z}_+} \Pi_n$ under the identification of ν_ℓ 's with multiplicities of ℓ 's, respectively, leading to a partition $\lambda = (1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \dots)$ of an integer $N_\lambda = \sum_{\ell=1}^{\infty} \ell \nu_\ell \in \mathbb{Z}_+$.

Let $c_0 = 1, c_1, c_2, \dots$ be a sequence of non-negative numbers such that not all c_k 's vanish for $k \geq 1$, and assume that the corresponding generating function

$$f(s) := 1 + \sum_{k=1}^{\infty} c_k s^k \quad (2.1)$$

is finite for all $|s| < 1$. For $z \in (0, 1)$, let us define a probability measure Q_z on the space $\Phi = \mathbb{Z}_+^{\mathbb{N}}$ as the distribution of a random sequence $\{\nu_\ell, \ell \in \mathbb{N}\}$ with mutually independent values and marginal distributions

$$Q_z\{\nu_\ell = k\} = \frac{c_k z^{k\ell}}{f(z^\ell)}, \quad k \in \mathbb{Z}_+. \quad (2.2)$$

Lemma 2.1. *For each $z \in (0, 1)$, the condition*

$$F(z) := \prod_{\ell=1}^{\infty} f(z^\ell) < \infty \quad (2.3)$$

is necessary and sufficient in order that $Q_z(\Phi_0) = 1$. Furthermore, if $f(s)$ is finite for all $|s| < 1$ then condition (2.3) is satisfied.

Proof. According to (2.2), $Q_z\{\nu_\ell > 0\} = 1 - f(z^\ell)^{-1}$ ($\ell \in \mathbb{N}$). Hence, Borel–Cantelli's lemma implies that $Q_z\{\nu \in \Phi_0\} = 1$ if and only if $\sum_{\ell=1}^{\infty} (1 - f(z^\ell)^{-1}) < \infty$. In turn, the latter inequality is equivalent to (2.3). To prove the second statement, observe using (2.1) that

$$\begin{aligned} \ln F(z) &= \sum_{\ell=1}^{\infty} \ln f(z^\ell) \leq \sum_{\ell=1}^{\infty} (f(z^\ell) - 1) = \sum_{k=1}^{\infty} c_k \sum_{\ell=1}^{\infty} z^{k\ell} \\ &= \sum_{k=1}^{\infty} \frac{c_k z^k}{1 - z^k} \leq \frac{1}{1 - z} \sum_{k=1}^{\infty} c_k z^k \leq \frac{f(z)}{1 - z} < \infty, \end{aligned}$$

which implies (2.3). □

Lemma 2.1 ensures that the random sequence $\{\nu_\ell\}$ belongs (Q_z -a.s.) to the space Φ_0 and therefore determines a (random) finite partition $\lambda \in \Pi$. By the mutual independence of the values ν_ℓ , the corresponding Q_z -probability is given by

$$Q_z(\lambda) = \prod_{\ell=1}^{\infty} \frac{c_{\nu_\ell} z^{\ell \nu_\ell}}{f(z^\ell)} = \frac{c(\lambda) z^{N_\lambda}}{F(z)}, \quad \lambda \in \Pi, \quad (2.4)$$

where $N_\lambda := \sum_{\ell=1}^{\infty} \ell \nu_\ell$ and $c(\lambda) := \prod_{\ell=1}^{\infty} c_{\nu_\ell} < \infty$ (cf. (1.6)).

Remark 2.1. The infinite product defining $c(\lambda)$ contains only finitely many terms different from 1 (since c_{ν_ℓ} is reduced to $c_0 = 1$ for $\ell \notin \text{supp } \nu$).

Remark 2.2. For the “empty” partition $\lambda_\emptyset \vdash 0$ formally associated with the configuration $\nu \equiv 0$, formula (2.4) yields $Q_z(\lambda_\emptyset) = F(z)^{-1} > 0$. On the other hand, $Q_z(\lambda_\emptyset) < 1$, since $f(s) > f(0) = 1$ for $s > 0$ and hence, according to definition (2.3), $F(z) > 1$.

On the subspace $\Pi_n \subset \Pi$, the measure Q_z induces the conditional distribution

$$P_n(\lambda) := Q_z(\lambda | \Pi_n) = \frac{Q_z(\lambda)}{Q_z(\Pi_n)}, \quad \lambda \in \Pi_n. \quad (2.5)$$

Formula (2.5) is well defined as long as $Q_z(\Pi_n) > 0$, that is, there is at least one partition $\lambda \in \Pi_n$ with $c(\lambda) > 0$ (see (2.4)). An obvious sufficient condition is as follows.

Lemma 2.2. *Suppose that $c_1 > 0$. Then $Q_z(\Pi_n) > 0$ for all $n \in \mathbb{Z}_+$.*

The following key fact explains the specific form of definition (2.2).

Lemma 2.3. *Formula (2.5) for the measure P_n is reduced to expression (1.6); in particular, P_n does not depend on z .*

Proof. If $\Pi_n \ni \lambda \leftrightarrow \nu \in \Phi_0$ then $N_\lambda = n$ and hence formula (2.4) is reduced to $Q_z(\lambda) = c(\lambda)z^n / F(z)$. In turn, the ratio in (2.5) amounts to expression (1.6), which is z -free. \square

2.2. A class of measures Q_z

Recalling expansion (2.1) for the generating function $f(s)$, consider the corresponding expansion of its logarithm,

$$H(s) := \ln f(s) = \sum_{k=1}^{\infty} a_k s^k, \quad |s| < 1. \quad (2.6)$$

Remark 2.3. Substituting expansion (2.1) into (2.6), it is clear that $a_1 = c_1$; more generally, if $j_* := \min\{j \geq 1 : c_j > 0\}$ and $k_* := \min\{k \geq 1 : a_k \neq 0\}$ then $j_* = k_*$ and $c_{j_*} = a_{k_*}$.

Under the measure Q_z defined in (2.2), the probability generating function $g_\nu(s; \ell) := E_z(s^{\nu_\ell})$ is given by

$$g_\nu(s; \ell) = \frac{f(sz^\ell)}{f(z^\ell)}, \quad |s| \leq 1 \quad (2.7)$$

(for notational simplicity, we suppress the dependence on z , which should cause no confusion), and so, using (2.6), its logarithm is expanded as

$$\ln g_\nu(s; \ell) = H(sz^\ell) - H(z^\ell) = \sum_{k=1}^{\infty} a_k (s^k - 1) z^{k\ell}, \quad |s| \leq 1. \quad (2.8)$$

Likewise, the characteristic function $\varphi_\nu(t; \ell) := E_z(e^{it\nu_\ell})$ is given by

$$\varphi_\nu(t; \ell) = \frac{f(z^\ell e^{it})}{f(z^\ell)}, \quad t \in \mathbb{R}, \quad (2.9)$$

and the principal branch of its logarithm (corresponding to $\ln \varphi_\nu(0; \ell) = 0$) is represented as

$$\ln \varphi_\nu(t; \ell) = \sum_{k=1}^{\infty} a_k (e^{ikt} - 1) z^{k\ell}, \quad t \in \mathbb{R}. \quad (2.10)$$

For $q \in \mathbb{N}$, denote by $m_q(\ell) := E_z(\nu_\ell^q)$ the moments of ν_ℓ , and let $\varkappa_q(\ell)$ be the cumulants of ν_ℓ , with the exponential generating function

$$\ln g_\nu(e^t; \ell) = \sum_{q=1}^{\infty} \varkappa_q(\ell) \frac{t^q}{q!}. \quad (2.11)$$

Substituting (2.8) into (2.11) and Taylor expanding the exponential function, we get

$$\ln g_\nu(e^t; \ell) = \sum_{k=1}^{\infty} a_k (e^{kt} - 1) z^{k\ell} = \sum_{q=1}^{\infty} \frac{t^q}{q!} \sum_{k=1}^{\infty} k^q a_k z^{k\ell},$$

and by a comparison with (2.11) it follows that

$$\varkappa_q(\ell) = \sum_{k=1}^{\infty} k^q a_k z^{k\ell}, \quad q \in \mathbb{N}. \quad (2.12)$$

In particular, from (2.12) we obtain the mean and variance of ν_ℓ ,

$$E_z(\nu_\ell) = m_1(\ell) = \varkappa_1(\ell) = \sum_{k=1}^{\infty} k a_k z^{k\ell}, \quad (2.13)$$

$$\text{Var}_z(\nu_\ell) = m_2(\ell) - m_1(\ell)^2 = \varkappa_2(\ell) = \sum_{k=1}^{\infty} k^2 a_k z^{k\ell}. \quad (2.14)$$

More generally, using a well-known recursion between the cumulants and moments (see, e.g., [15, Section 3.14])

$$m_q = \varkappa_q + \sum_{i=1}^{q-1} \binom{q-1}{i-1} \varkappa_i m_{q-i} \quad (2.15)$$

it is easy to see by a simple induction that the moments m_q ($q \in \mathbb{N}$) are expressed as linear combinations of the cumulants $\varkappa_1, \dots, \varkappa_q$ with positive (in fact, integer) coefficients, which gives, in view of (2.12),

$$m_q(\ell) = \sum_{i=1}^q C_{i,q} \varkappa_i(\ell) = \sum_{i=1}^q C_{i,q} \sum_{k=1}^{\infty} k^i a_k z^{k\ell}, \quad C_{i,q} > 0 \quad (i = 1, \dots, q). \quad (2.16)$$

Furthermore, using a rescaling relation $\varkappa_q[cX] = c^q \varkappa_q[X]$ and the additive property of cumulants for independent summands, we obtain the cumulants of the random variable $N_\lambda = \sum_{\ell=1}^{\infty} \ell \nu_\ell$,

$$\varkappa_q[N_\lambda] = \sum_{\ell=1}^{\infty} \ell^q \varkappa_q(\ell) = \sum_{\ell=1}^{\infty} \ell^q \sum_{k=1}^{\infty} k^q a_k z^{k\ell} \quad (2.17)$$

and, similarly to (2.16), the corresponding moments

$$E_z(N_\lambda^q) = \sum_{i=1}^q C_{i,q} \sum_{\ell=1}^{\infty} \ell^q \sum_{k=1}^{\infty} k^q a_k z^{k\ell}. \quad (2.18)$$

For $s \in \mathbb{C}$ such that $\sigma := \Re s > 0$, denote

$$A(s) := \sum_{k=1}^{\infty} \frac{a_k}{k^s}, \quad A^+(\sigma) := \sum_{k=1}^{\infty} \frac{|a_k|}{k^\sigma} \leq \infty. \quad (2.19)$$

Most of our results are valid under the condition $A^+(1) < \infty$, or sometimes $A^+(\eta) < \infty$ for some $\eta > 0$ (in particular, in Theorem 4.1). For a local limit theorem (see Theorem 6.1), we require an additional technical condition on the generating function $f(s)$.

Assumption 2.1. The coefficients (a_k) in a power series expansion (2.1) of $H(s) = \ln f(s)$ are such that $a_1 > 0$ and, for any $\theta \in (0, 1)$ and all $t \in \mathbb{R}$, the following inequality holds, with some constant $\delta_1 > 0$,

$$\sum_{k=1}^{\infty} a_k \theta^k (1 - \cos kt) \geq \delta_1 a_1 \theta (1 - \cos t). \quad (2.20)$$

Remark 2.4. Assumption 2.1 is obviously satisfied (with $\delta_1 = 1$) when *all* a_k are positive.

Due to Remark 2.3, the condition $a_1 > 0$ is equivalent to $c_1 > 0$. Moreover, from (2.9) and (2.10) we note that

$$\ln |\varphi_\nu(t; \ell)| = \frac{1}{2} \ln \frac{f(z^\ell e^{it}) f(z^\ell e^{-it})}{f(z^\ell)^2} = - \sum_{k=1}^{\infty} a_k z^{k\ell} (1 - \cos kt), \quad (2.21)$$

hence, condition (2.20) can be equivalently rewritten (for any $\theta \in (0, 1)$ and all $t \in \mathbb{R}$) as

$$\frac{1}{2} \ln \frac{f(\theta e^{it}) f(\theta e^{-it})}{f(\theta)^2} \leq -\delta_1 c_1 \theta (1 - \cos t). \quad (2.22)$$

2.3. Examples

Let us now consider a few illustrative examples. The first three are well known in the theory of decomposable combinatorial structures, corresponding, respectively, to the three well-known meta-classes — multisets, selections and assemblies (see [1, 2, 12]. More specifically, Example 2.1 below corresponds to the ensemble of weighted partitions, including the case of unrestricted partitions under the uniform (equiprobable) distribution; Example 2.2 leads to (weighted) partitions with bounds on the multiplicities of parts, including the case of uniform partitions with distinct parts; Example 2.3 corresponds to partitions representing the cycle structure of permutations. To the best of our knowledge, Example 2.4 appears to be new.

Example 2.1 (multisets). For $r \in (0, \infty)$, $\rho \in (0, 1]$, let Q_z be a measure determined by formula (2.2) with coefficients

$$c_k := \binom{r+k-1}{k} = \frac{r(r+1) \cdots (r+k-1)}{k!}, \quad k \in \mathbb{Z}_+. \quad (2.23)$$

In particular, $c_0 = 1$ and $c_1 = r\rho > 0$. By the binomial expansion formula, the generating function of sequence (2.23) is given by

$$f(s) = (1 - \rho s)^{-r}, \quad |s| < \rho^{-1}, \quad (2.24)$$

and formula (2.2) specializes to

$$Q_z\{\nu_\ell = k\} = \binom{r+k-1}{k} \rho^k z^{k\ell} (1 - \rho z^\ell)^r, \quad k \in \mathbb{Z}_+, \quad (2.25)$$

which is a negative binomial distribution with parameters r and $p = 1 - \rho z^\ell$.

If $r = 1$ then $c_k = \rho^k$, $f(s) = (1 - \rho s)^{-1}$ and, according to (2.25),

$$Q_z\{\nu_\ell = k\} = \rho^k z^{k\ell} (1 - \rho z^\ell), \quad k \in \mathbb{Z}_+.$$

In turn, from (1.6) we get

$$P_n(\lambda) = \mathfrak{C}_n^{-1} \rho^{Y_\lambda(0)} \quad (\lambda \in \Pi_n), \quad \mathfrak{C}_n = \sum_{\lambda \in \Pi_n} \rho^{Y_\lambda(0)} \quad (2.26)$$

where $Y_\lambda(0) = \sum_{\ell=1}^{\infty} \nu_\ell$ is the “length” of partition λ (see Section 1). Furthermore, if also $\rho = 1$ then (2.26) is reduced to the uniform distribution on Π_n (see (1.6)),

$$P_n(\lambda) = \frac{1}{\#(\Pi_n)}, \quad \lambda \in \Pi_n.$$

In the general case, using (2.24) we note that

$$\ln f(s) = -r \ln(1 - \rho s) = r \sum_{k=1}^{\infty} \frac{\rho^k s^k}{k},$$

and so the coefficients (a_k) in expansion (2.6) are given by

$$a_k = \frac{r \rho^k}{k} > 0, \quad k \in \mathbb{N}. \quad (2.27)$$

As pointed out in Remark 2.4, this implies that Assumption 2.1 is satisfied; also, it readily follows that $A^+(\sigma) < \infty$ for any $\sigma > 0$.

Example 2.2 (selections). For $r \in \mathbb{N}$, $\rho \in (0, 1]$, consider the generating function

$$f(s) = (1 + \rho s)^r, \quad |s| < \rho^{-1}, \quad (2.28)$$

with the coefficients in expansion (2.1) given by

$$c_k = \binom{r}{k} \rho^k = \frac{r(r-1) \cdots (r-k+1)}{k!} \rho^k, \quad k = 0, 1, \dots, r. \quad (2.29)$$

In particular, $c_0 = 1$, $c_1 = r\rho > 0$. Accordingly, formula (2.2) gives a binomial distribution

$$Q_z\{\nu_\ell = k\} = \binom{r}{k} \frac{\rho^k z^{k\ell}}{(1 + \rho z^\ell)^r}, \quad k = 0, 1, \dots, r, \quad (2.30)$$

with parameters r and $p = \rho z^\ell (1 + \rho z^\ell)^{-1}$. From (2.28) we obtain

$$\ln f(s) = r \ln(1 + \rho s) = r \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \rho^k}{k} s^k,$$

hence the coefficients (a_k) in expansion (2.6) are given by

$$a_k = \frac{r(-1)^{k-1} \rho^k}{k}, \quad k \in \mathbb{N}, \quad (2.31)$$

and in particular $a_1 = r\rho > 0$. Note that $A^+(\sigma) < \infty$ for any $\sigma > 0$.

In the special case $r = 1$, the measure Q_z is concentrated on the subspace $\check{\Pi} \subset \Pi$ of partitions with distinct parts (i.e., where any $\ell \in \mathbb{N}$ is involved no more than once). Here we have $c_0 = 1$, $c_1 = \rho$ and $c_k = 0$ ($k \geq 2$), so that (2.30) is reduced to

$$Q_z\{\nu_\ell = k\} = \frac{\rho^k z^{k\ell}}{1 + \rho z^\ell}, \quad k = 0, 1 \quad (\ell \in \mathbb{N}).$$

Accordingly, formula (1.6) specifies on $\check{\Pi}_n$ the distribution

$$P_n(\lambda) = \mathfrak{C}_n^{-1} \rho^{Y_\lambda(0)} \quad (\lambda \in \check{\Pi}_n), \quad \mathfrak{C}_n = \sum_{\lambda \in \check{\Pi}_n} \rho^{Y_\lambda(0)}, \quad (2.32)$$

where $Y_\lambda(0) = \sum_{\ell=1}^{\infty} \nu_\ell$ (cf. Example 2.1). Furthermore, if also $\rho = 1$ then (2.32) is reduced to the uniform distribution on $\check{\Pi}_n$,

$$P_n(\lambda) = \frac{1}{\#(\check{\Pi}_n)}, \quad \lambda \in \check{\Pi}_n.$$

Finally, let us check that Assumption 2.1 holds (with $\delta_1 = (1 + \rho)^{-2}$). It is more convenient to use version (2.22). Substituting (2.28) and recalling that $c_1 = r\rho > 0$, we obtain

$$\begin{aligned} \frac{1}{2} \ln \left(\frac{f(\theta e^{it}) f(\theta e^{-it})}{f(\theta)^2} \right) &= \frac{r}{2} \ln \left(\frac{1 + 2\rho\theta \cos t + \rho^2\theta^2}{(1 + \rho\theta)^2} \right) \\ &\leq \frac{r}{2} \left(\frac{1 + 2\rho\theta \cos t + \rho^2\theta^2}{(1 + \rho\theta)^2} - 1 \right) \leq -\frac{c_1\theta(1 - \cos t)}{(1 + \rho)^2}. \end{aligned}$$

Example 2.3 (assemblies). For $r \in (0, \infty)$, $\rho \in [0, 1]$, consider the generating function

$$f(s) = \exp \left(\frac{rs}{1 - \rho s} \right) = \exp \left(r \sum_{k=1}^{\infty} s^k \rho^{k-1} \right), \quad |s| < \rho^{-1}. \quad (2.33)$$

Clearly, the corresponding coefficients c_k in expansion (2.1) are positive, with $c_0 = 1$, $c_1 = r$, $c_2 = \frac{1}{2}r^2 + r\rho$, etc.; more systematically, one can use the well-known Faà di Bruno's formula generalizing the chain rule to higher derivatives (see, e.g., [14, Ch. I, §12, p. 34]) to obtain

$$c_k = \rho^k \sum_{m=1}^k \left(\frac{r}{\rho} \right)^m \sum_{(j_1, \dots, j_k) \in \mathcal{J}_m} \frac{1}{j_1! \cdots j_k!}, \quad k \in \mathbb{N}, \quad (2.34)$$

where \mathcal{J}_m is the set of all non-negative integer k -tuples (j_1, \dots, j_k) such that $j_1 + \cdots + j_k = m$ and $1 \cdot j_1 + 2 \cdot j_2 + \cdots + k \cdot j_k = k$.

Remark 2.5. Note that the k -tuples $(j_1, \dots, j_k) \in \mathcal{J}_m$ are in a one-to-one correspondence with partitions of k involving precisely m different integers as parts, where an element j_ℓ has the meaning of the multiplicity of part $\ell \in \mathbb{N}$.

Taking the logarithm of (2.33), we see that the coefficients in expansion (2.6) are given by

$$a_k = r\rho^{k-1} > 0, \quad k \in \mathbb{N}. \quad (2.35)$$

Therefore, Assumption 2.1 is automatic; moreover, $A^+(\sigma) < \infty$ for any $\sigma > 0$, except for the case $\rho = 1$ where $A^+(\sigma) < \infty$ only for $\sigma > 1$.

In the particular case $\rho = 0$, we have $f(s) = e^{rs}$ and so expression (2.34) is replaced by $c_k = r^k/k!$, whereas (2.35) simplifies to $a_1 = r$ and $a_k = 0$ for $k \geq 2$. The random variables ν_ℓ have a Poisson distribution with parameter rz^ℓ ,

$$Q_z\{\nu_\ell = k\} = \frac{r^k z^{k\ell}}{k!} e^{-rz^\ell}, \quad k \in \mathbb{Z}_+,$$

which leads, according to (1.6), to the distribution on Π_n of the following form

$$P_n(\lambda) = \mathfrak{C}_n^{-1} \prod_{\ell=1}^{\infty} \frac{r^{\nu_\ell}}{\nu_\ell!}, \quad \mathfrak{C}_n = \sum_{\{\nu_\ell\} \leftrightarrow \lambda \in \Pi_n} \prod_{\ell=1}^{\infty} \frac{r^{\nu_\ell}}{\nu_\ell!}.$$

Example 2.4. Let $r \in (0, \infty)$, $\rho \in (0, 1]$, and consider the generating function

$$f(s) = \left(\frac{-\ln(1 - \rho s)}{\rho s} \right)^r = \left(1 + \sum_{k=1}^{\infty} \frac{\rho^k s^k}{k+1} \right)^r =: f_1(s)^r. \quad (2.36)$$

From (2.36) it is clear that $c_0 = 1$, $c_1 = \frac{1}{2}r\rho > 0$ and, more generally, all $c_k > 0$. Let us analyze the coefficients (a_k) in the power series expansion of $H(s) = r \ln f_1(s)$ (see (2.6)). Differentiation of this identity with respect to s gives

$$r f_1'(s) = f_1(s) \sum_{k=1}^{\infty} k a_k s^{k-1}. \quad (2.37)$$

Differentiating (2.37) further m times ($m \geq 0$), by the Leibniz rule we obtain

$$f_1^{(m+1)}(s) = \frac{1}{r} \sum_{j=0}^m \binom{m}{j} f_1^{(m-j)}(s) \sum_{k=j+1}^{\infty} \frac{k!}{(k-j-1)!} a_k s^{k-j-1},$$

and in particular

$$f_1^{(m+1)}(0) = \frac{1}{r} \sum_{j=0}^m \binom{m}{j} f_1^{(m-j)}(0) (m+1)! a_{m+1}. \quad (2.38)$$

But we know from (2.36) that $f_1^{(j)}(0) = \rho^j j! / (j+1)$, so (2.38) specializes to the equation

$$\frac{\rho^{m+1} (m+1)!}{m+2} = \frac{1}{r} \sum_{j=0}^m \frac{m!}{j! (m-j)!} \cdot \frac{\rho^{m-j} (m-j)!}{m-j+1} (j+1)! a_{j+1},$$

or, after some cancellations,

$$\frac{m+1}{m+2} = \frac{1}{r} \sum_{j=0}^m \frac{\rho^{-j-1}(j+1)}{m-j+1} a_{j+1}. \quad (2.39)$$

Denoting for short $\tilde{a}_j := r^{-1} \rho^{-j} j a_j$, equation (2.39) simplifies to

$$\frac{m+1}{m+2} = \frac{\tilde{a}_1}{m+1} + \frac{\tilde{a}_2}{m} + \cdots + \frac{\tilde{a}_m}{2} + \tilde{a}_{m+1}. \quad (2.40)$$

Setting here $m = 0, 1, 2, 3, \dots$ we can in principle find successively all \tilde{a}_m ,

$$\tilde{a}_1 = \frac{1}{2}, \quad \tilde{a}_2 = \frac{5}{12}, \quad \tilde{a}_3 = \frac{3}{8}, \quad \tilde{a}_4 = \frac{251}{720}, \dots,$$

but the fractions quickly become quite cumbersome. However, it is not hard to obtain suitable estimates of \tilde{a}_m . Observe that (2.40) implies

$$\frac{m+1}{m+2} \leq \frac{\tilde{a}_1}{m} + \frac{\tilde{a}_2}{m-1} + \cdots + \tilde{a}_m + \tilde{a}_{m+1} = \frac{m}{m+1} + \tilde{a}_{m+1},$$

and it follows that

$$\tilde{a}_{m+1} \geq \frac{m+1}{m+2} - \frac{m}{m+1} = \frac{1}{(m+1)(m+2)} > 0,$$

or explicitly

$$a_{m+1} \geq \frac{r \rho^{m+1}}{(m+1)^2(m+2)} > 0. \quad (2.41)$$

On the other hand, from (2.40) we get

$$\tilde{a}_{m+1} = \frac{m+1}{m+2} - \frac{\tilde{a}_1}{m+1} - \frac{\tilde{a}_2}{m} - \cdots - \frac{\tilde{a}_m}{2} \leq \frac{m+1}{m+2} - \frac{\tilde{a}_1}{m+1},$$

hence

$$\tilde{a}_{m+1} \leq \frac{m+1}{m+2} - \frac{1/2}{m+1} = \frac{2m^2 + 3m}{2(m+1)(m+2)}$$

and therefore

$$a_{m+1} \leq \frac{r \rho^{m+1}(2m^2 + 3m)}{2(m+1)^2(m+2)}. \quad (2.42)$$

As a result, combining (2.41) and (2.42) we obtain, for all $k \in \mathbb{N}$,

$$\frac{r \rho^k}{k^2(k+1)} \leq a_k \leq \frac{r \rho^k(2k^2 - k - 1)}{2k^2(k+1)} \leq \frac{r \rho^k}{k+1}.$$

In particular, this implies that $A^+(\sigma) < \infty$ for any $\sigma > 0$; furthermore, since all $a_k > 0$ it follows that Assumption 2.1 is automatically satisfied.

3. Asymptotics of the expectation

3.1. Calibration of the parameter z

We want to find a suitable parameter $z \in (0, 1)$ subject to the asymptotic condition

$$E_z(N_\lambda) \sim n \quad (n \rightarrow \infty), \quad (3.1)$$

where $N_\lambda = \sum_{\ell=1}^{\infty} \ell \nu_\ell$ and E_z denotes expectation with respect to Q_z . Set

$$z = e^{-\alpha_n}, \quad \alpha_n = \kappa n^{-1/2}, \quad (3.2)$$

where the constant $\kappa > 0$ is to be fitted to ensure (3.1). Hence, recalling (2.13), we get

$$E_z(N_\lambda) = \sum_{k=1}^{\infty} k a_k \sum_{\ell=1}^{\infty} \ell e^{-k\alpha_n \ell} = \sum_{k=1}^{\infty} \frac{k a_k e^{-k\alpha_n}}{(1 - e^{-k\alpha_n})^2}. \quad (3.3)$$

Theorem 3.1. *Suppose that $A^+(1) < \infty$ (see (2.19)), and set*

$$\kappa := \sqrt{A(1)}. \quad (3.4)$$

Then condition (3.1) is satisfied.

Proof. Note that for any $b > 0$, $\theta > 0$, there is a global bound

$$\frac{e^{-\theta t}}{(1 - e^{-t})^b} \leq C t^{-b}, \quad t > 0, \quad (3.5)$$

with some constant $C = C(b, \theta) > 0$. Therefore, the general summand in (3.3) is bounded, uniformly in k , by $\alpha_n^{-2} O(|a_k| k^{-1})$, where $\sum_{k=1}^{\infty} |a_k| k^{-1} = A^+(1) < \infty$. Hence, by Lebesgue's dominated convergence theorem one can pass to the limit in (3.3) termwise to obtain

$$\lim_{n \rightarrow \infty} n^{-1} E_z(N_\lambda) = \frac{1}{\kappa^2} \sum_{k=1}^{\infty} \frac{a_k}{k} = \frac{A(1)}{\kappa^2} = 1, \quad (3.6)$$

according to (3.2) and (3.4). □

The following expression for $A(1)$ directly in terms of the generating function $H(s)$ is sometimes useful (e.g., for computer calculations of the coefficient κ ; cf. also Theorem 1.1).

Lemma 3.2. *If $A^+(1) < \infty$ then*

$$A(1) = \int_0^1 s^{-1} H(s) \, ds. \quad (3.7)$$

Proof. Using expansion (2.6), for any $q \in (0, 1)$ we immediately obtain

$$\int_0^q s^{-1} H(s) \, ds = \sum_{k=1}^{\infty} a_k \int_0^q s^{k-1} \, ds = \sum_{k=1}^{\infty} \frac{a_k q^k}{k} \rightarrow \sum_{k=1}^{\infty} \frac{a_k}{k} = A(1)$$

as $q \uparrow 1$, and the lemma is proved. □

Assumption 3.1. Throughout the rest of the paper, we assume that the parameter z is chosen according to formulas (3.2), (3.4). In particular, the measure Q_z becomes dependent on n , as well as the Q_z -probabilities and the corresponding expected values.

3.2. Asymptotics of the mean Young diagram

Recalling that the upper boundary $Y_\lambda(x)$ of the Young diagram Υ_λ is given by formula (1.1), we obtain, similarly to (3.3),

$$E_z[Y_\lambda(x)] = \sum_{k=1}^{\infty} k a_k \sum_{\ell > x} e^{-k\alpha_n \ell}, \quad x \geq 0. \quad (3.8)$$

Theorem 3.3. *Uniformly in $x \in [\delta, \infty)$, for any $\delta > 0$,*

$$\lim_{n \rightarrow \infty} n^{-1/2} E_z[Y_\lambda(x n^{1/2})] = \omega^*(x), \quad (3.9)$$

where the limit shape function $\omega^*(x)$ is defined in (1.7).

Proof. Setting $\ell_n^* \equiv \ell_n^*(x) := 1 + \lfloor x n^{1/2} \rfloor \equiv 1 + \max\{\ell \in \mathbb{N} : \ell \leq x n^{1/2}\}$ and recalling that $\alpha_n = \kappa n^{-1/2}$ (see (3.2)), from (3.8) we obtain

$$\kappa n^{-1/2} E_z[Y_\lambda(x n^{1/2})] = \alpha_n \sum_{\ell=\ell_n^*}^{\infty} \sum_{k=1}^{\infty} k a_k e^{-k\alpha_n \ell} = \alpha_n \sum_{\ell=\ell_n^*}^{\infty} \phi(\alpha_n \ell), \quad (3.10)$$

where (see (2.6))

$$\phi(t) := \sum_{k=1}^{\infty} k a_k e^{-kt} = e^{-t} H'(e^{-t}), \quad t > 0.$$

The right-hand side of (3.10) can be viewed as a Riemann integral sum for the function $\phi(t)$ (over an infinite interval $t \in (\kappa x, \infty)$ and with the mesh size α_n), suggesting its convergence, as $n \rightarrow \infty$, to the corresponding integral. More precisely, using Euler–Maclaurin’s summation formula (see, e.g., [5, §12.2]) we have

$$\sum_{\ell=\ell_n^*}^{\infty} \phi(\alpha_n \ell) = \int_{\ell_n^*}^{\infty} \phi(\alpha_n t) dt + \alpha_n \int_{\ell_n^*}^{\infty} (t - [t] - 1) \phi'(\alpha_n t) dt. \quad (3.11)$$

Note that, uniformly in $x \geq \delta$,

$$\begin{aligned} \left| \int_{\ell_n^*}^{\infty} (t - [t] - 1) \phi'(\alpha_n t) dt \right| &\leq \int_{\ell_n^*}^{\infty} |\phi'(\alpha_n t)| dt \\ &= \alpha_n^{-1} \int_0^{e^{-\alpha_n \ell_n^*}} |H'(s) + s H''(s)| ds = O(\alpha_n^{-1}), \end{aligned} \quad (3.12)$$

since $\alpha_n \ell_n^* \rightarrow \kappa x$ and both $H'(s)$ and $H''(s)$ are bounded on any interval $s \in [0, q]$ with $q < 1$. On the other hand, again uniformly in $x \geq \delta$,

$$\int_{\ell_n^*}^{\infty} \phi(\alpha_n t) dt = \alpha_n^{-1} \int_0^{e^{-\alpha_n \ell_n^*}} H'(s) ds \sim \alpha_n^{-1} H(e^{-\kappa x}). \quad (3.13)$$

Hence, substituting (3.12) and (3.13) into (3.11), and returning to (3.10), we obtain

$$\lim_{n \rightarrow \infty} n^{-1/2} E_z[Y_\lambda(x n^{1/2})] = \kappa^{-1} H(e^{-\kappa x}) \equiv \omega^*(x),$$

where, according to the above, the convergence is uniform for $x \geq \delta > 0$. \square

4. Refined asymptotics of the expectation

We need to sharpen the asymptotics $E_z(N_\lambda) - n = o(n)$ provided by Theorem 3.1 (see (3.1)).

Theorem 4.1. *Under the condition $A^+(\eta) < \infty$ with some $\eta > 0$, we have*

$$E_z(N_\lambda) - n = O(n^{1/2+\eta/2}), \quad n \rightarrow \infty.$$

For the proof of Theorem 4.1, some preparations are required. Let a function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be such that $f \in C^1(\mathbb{R}_+)$, $f(0) = 0$ and f, f' are (absolutely) integrable on \mathbb{R}_+ . Set

$$F(h) := \sum_{\ell=1}^{\infty} f(h\ell), \quad h > 0 \quad (4.1)$$

(using Euler–Maclauren’s summation formula similar to (3.11), one can verify that the above conditions on f ensure convergence of series (4.1)), and assume that for some $\beta > 1$

$$F(h) = O(h^{-\beta}), \quad h \rightarrow \infty. \quad (4.2)$$

Let us also observe that $F(h) = o(1)$ as $h \rightarrow 0+$. Consider the Mellin transform of $F(h)$ (see, e.g., [27, Ch. VI, §9]),

$$\widehat{F}(s) := \int_0^\infty h^{s-1} F(h) dh = \zeta(s) \int_0^\infty x^{s-1} f(x) dx, \quad 1 < \Re s < \beta, \quad (4.3)$$

where $\zeta(s) = \sum_{k=1}^\infty k^{-s}$ is the Riemann zeta function. By the well-known properties of $\zeta(s)$ (see, e.g., [21, §2.1, p. 13]), from (4.2) and (4.3) it follows that the function $\widehat{F}(s)$ is meromorphic in the strip $0 < \Re s < \beta$, with a single pole at $s = 1$. Set

$$\Delta_f(h) := F(h) - \frac{1}{h} \int_0^\infty f(x) dx, \quad h > 0. \quad (4.4)$$

Then the Müntz lemma (see [21, § 2.11, pp. 28–29])) gives

$$\widehat{F}(s) = \int_0^\infty h^{s-1} \Delta_f(h) dh, \quad 0 < \Re s < 1,$$

and the Mellin transform inversion formula [27, Ch. VI, §9, Theorem 9a] implies

$$\Delta_f(h) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h^{-s} \widehat{F}(s) ds, \quad 0 < c < 1. \quad (4.5)$$

Proof of Theorem 4.1. Denote $f(x) := x e^{-\alpha_n x}$, then (4.1) specializes to

$$F(h) = h \sum_{\ell=1}^{\infty} \ell e^{-\alpha_n h \ell} = \frac{h e^{-\alpha_n h}}{(1 - e^{-\alpha_n h})^2}, \quad h > 0.$$

According to (3.3) we have

$$E_z(N_\lambda) = \sum_{k=1}^{\infty} a_k F(k). \quad (4.6)$$

Note that

$$\frac{1}{k} \int_0^\infty f(x) dx = \frac{1}{k} \int_0^\infty x e^{-\alpha_n x} dx = \frac{1}{k \alpha_n^2}.$$

Moreover, recalling (3.2) and (3.4) (see (2.19)) we have

$$\sum_{k=1}^\infty \frac{a_k}{k \alpha_n^2} = \frac{n A(1)}{\kappa^2} \equiv n. \quad (4.7)$$

Hence, subtracting (4.7) from (4.6) and recalling notation (4.4), we obtain the representation

$$E_z(N_\lambda) - n = \sum_{k=1}^\infty a_k \Delta_f(k). \quad (4.8)$$

Clearly, the functions f and F satisfy the above hypotheses (with $\beta = \infty$). Furthermore, from (4.3) we easily obtain the Mellin transform

$$\widehat{F}(s) = \zeta(s) \int_0^\infty h^s e^{-\alpha_n h} dh = \alpha_n^{-s-1} \zeta(s) \Gamma(s+1), \quad 1 < \Re s < \infty, \quad (4.9)$$

where $\Gamma(\cdot)$ is the gamma function. Since $\Gamma(s)$ is analytic for $\Re s > 0$ (cf. [20, §4.41, p. 148]) and, as already mentioned, $\zeta(s)$ has a single (simple) pole at point $s = 1$, it follows that expression (4.9) is meromorphic in the half-plane $\Re s > -1$, with a single pole at $s = 1$, and hence it provides an analytic continuation of the function $\widehat{F}(s)$ into the strip $-1 < \Re s < 1$.

Using (4.5) and (4.9), and recalling formula (2.19), we can rewrite (4.8) as

$$\begin{aligned} E_z(N_\lambda) - n &= \frac{1}{2\pi i} \sum_{k=1}^\infty a_k \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s) \Gamma(s+1)}{\alpha_n^{s+1} k^s} ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{A(s) \zeta(s) \Gamma(s+1)}{\alpha_n^{s+1}} ds \quad (0 < c < 1). \end{aligned} \quad (4.10)$$

Let us show that the integration contour $\Re s = c$ in (4.10) can be moved to $\Re s = \eta \in (0, 1)$ (see the hypotheses of the theorem). By the Cauchy theorem, it suffices to check that

$$\lim_{t \rightarrow \infty} \int_{\eta-it}^{c+it} \frac{A(s) \zeta(s) \Gamma(s+1)}{\alpha_n^{s+1}} ds = 0. \quad (4.11)$$

To this end, note that for $s = \sigma + it$ with $\eta \leq \sigma \leq c < 1$

$$|A(s)| \leq A^+(\eta) < \infty, \quad |\alpha_n^{-s-1}| \leq \alpha_n^{-c-1}. \quad (4.12)$$

Furthermore, we have the following asymptotic estimates as $t \rightarrow \infty$, uniform in the strip $\Re s \in [0, 1]$ (see [20, §4.42, p. 151] and [13, Theorem 1.9, p. 25], respectively)

$$\Gamma(s+1) = O(|t|^{\sigma+1/2} e^{-\pi|t|/2}), \quad \zeta(s) = O(|t|^{(1-\sigma)/2} \ln |t|). \quad (4.13)$$

By virtue of estimates (4.12) and (4.13), the integral in (4.11) is bounded by

$$O(1) A^+(\eta) \alpha_n^{-c-1} (c - \eta) |t|^{1+c/2} e^{-\pi|t|/2} \ln |t| \rightarrow 0 \quad (t \rightarrow \infty),$$

whence (4.11) follows. Hence, representation (4.10) takes the form

$$\begin{aligned} E_z(N_\lambda) - n &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{A(\eta + it) \zeta(\eta + it) \Gamma(\eta + 1 + it)}{\alpha_n^{\eta+1+it}} dt \\ &= \frac{O(1)}{\alpha_n^{1+\eta}} \int_{-\infty}^{\infty} A^+(\eta) |t|^{1+\eta/2} e^{-\pi|t|/2} \ln |t| dt = O(\alpha_n^{-1-\eta}) = O(n^{1/2+\eta/2}), \end{aligned}$$

according to (3.2). Thus, the proof of Theorem 4.1 is complete. \square

5. Asymptotics of higher-order moments

Throughout this section, we assume that $A^+(1) < \infty$ (see (2.19)).

5.1. The variance

Let $\sigma_z^2 := \text{Var}_z(N_\lambda)$ be the variance (with respect to the measure Q_z) of the random variable $N_\lambda = \sum_{\ell=1}^{\infty} \ell \nu_\ell$. Recalling that $\{\nu_\ell\}$ are mutually independent and using (2.14), we see that

$$\sigma_z^2 = \sum_{\ell=1}^{\infty} \ell^2 \text{Var}_z(\nu_\ell) = \sum_{\ell=1}^{\infty} \ell^2 \sum_{k=1}^{\infty} k^2 a_k z^{k\ell}. \quad (5.1)$$

Lemma 5.1. *As $n \rightarrow \infty$, we have $\sigma_z^2 \sim 2A(1) \kappa^{-3} n^{3/2}$.*

Proof. Substituting (3.2) into (5.1), we obtain

$$\sigma_z^2 = \sum_{k=1}^{\infty} k^2 a_k \sum_{\ell=1}^{\infty} \ell^2 e^{-k\alpha_n \ell} = \sum_{k=1}^{\infty} k^2 a_k \frac{e^{-k\alpha_n} (1 + e^{-k\alpha_n})}{(1 - e^{-k\alpha_n})^3}. \quad (5.2)$$

By estimate (3.5), the general term in series (5.2) is bounded by $\alpha_n^{-3} O(|a_k| k^{-1})$, uniformly in $k \in \mathbb{N}$. Hence, Lebesgue's dominated convergence theorem yields

$$\alpha_n^3 \sigma_z^2 \rightarrow 2 \sum_{k=1}^{\infty} \frac{a_k}{k} = 2A(1), \quad n \rightarrow \infty,$$

whence, in view of (3.2) and (3.4), the result follows. \square

We also need to analyze the variance of the function $Y_\lambda(x) = \sum_{\ell > x} \nu_\ell$, given by

$$\text{Var}_z[Y_\lambda(x)] = \sum_{\ell > x} \text{Var}_z(\nu_\ell) = \sum_{\ell > x} \sum_{k=1}^{\infty} k^2 a_k z^{k\ell}, \quad x \geq 0. \quad (5.3)$$

Observe that if $H(s) = \sum_{k=1}^{\infty} a_k s^k$ (see (2.6)) then $\sum_{k=1}^{\infty} k^2 a_k s^k = s^2 H''(s) + s H'(s)$. Hence, representation (5.3) can be rewritten as

$$\text{Var}_z[Y_\lambda(x)] = \sum_{\ell > x} (z^{2\ell} H''(z^\ell) + z^\ell H'(z^\ell)), \quad x \geq 0. \quad (5.4)$$

Lemma 5.2. *For any $\delta > 0$ we have, uniformly in $x \geq \delta$ as $n \rightarrow \infty$,*

$$\text{Var}_z[Y_\lambda(xn^{1/2})] = O(n^{1/2}).$$

Proof. Substituting (3.2) into (5.4), we obtain

$$\text{Var}_z[Y_\lambda(xn^{1/2})] = \sum_{\ell > xn^{1/2}} e^{-\alpha_n \ell} (e^{-\alpha_n \ell} H''(e^{-\alpha_n \ell}) + H'(e^{-\alpha_n \ell})). \quad (5.5)$$

Interpreting (5.5) as a Riemann integral sum and arguing as in the proof of Theorem 3.3, we conclude that equation (5.5) converges, uniformly in $x \geq \delta > 0$, to the limit

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\text{Var}_z[Y_\lambda(xn^{1/2})]}{n^{1/2}} &= \kappa^{-1} \int_{\kappa x}^{\infty} e^{-t} (e^{-t} H''(e^{-t}) + H'(e^{-t})) dt \\ &= \kappa^{-1} \int_0^{e^{-\kappa x}} (s H''(s) + H'(s)) ds \\ &= \kappa^{-1} \int_0^{e^{-\kappa x}} (s H'(s))' ds = \kappa^{-1} e^{-\kappa x} H'(e^{-\kappa x}), \end{aligned}$$

and since $H'(e^{-\kappa x}) = O(1)$ for $x \geq \delta > 0$, the claim of the lemma follows. \square

5.2. Auxiliary estimates

Denote $\nu_\ell^0 := \nu_\ell - E_z(\nu_\ell)$ ($\ell \in \mathbb{N}$), and consider the moments of order $q \in \mathbb{N}$

$$m_q(\ell) := E_z(\nu_\ell^q), \quad \mu_q(\ell) := E_z|\nu_\ell^0|^q. \quad (5.6)$$

Let us note a simple general inequality.

Lemma 5.3. *For each $q \geq 1$,*

$$\mu_q(\ell) \leq 2^q m_q(\ell), \quad \ell \in \mathbb{N}. \quad (5.7)$$

Proof. Applying the elementary inequality $(a + b)^q \leq 2^{q-1}(a^q + b^q)$ for any $a, b > 0$ and $q \geq 1$ (which follows from Hölder's inequality for the function $y = x^q$), we obtain

$$\mu_q(\ell) \leq E_z[(\nu_\ell + m_1(\ell))^q] \leq 2^{q-1}(m_q(\ell) + m_1(\ell)^q) \leq 2^q m_q(\ell),$$

where we also used Lyapunov's inequality $m_1(\ell)^q \leq m_q(\ell)$. \square

The following two lemmas are useful for estimation of higher-order moment sums.

Lemma 5.4. *For $q \in \mathbb{N}$, the function*

$$S_q(\theta) := \sum_{\ell=1}^{\infty} \ell^{q-1} e^{-\theta \ell}, \quad \theta > 0, \quad (5.8)$$

admits a representation

$$S_q(\theta) = \sum_{j=1}^q c_{j,q} \frac{e^{-\theta j}}{(1 - e^{-\theta})^j}, \quad \theta > 0, \quad (5.9)$$

with some constants $c_{j,q} > 0$ ($j = 1, \dots, q$); in particular, $c_{q,q} = (q-1)!$.

Proof. In the case $q = 1$, expression (5.8) is reduced to a geometric series

$$S_1(\theta) = \sum_{\ell=1}^{\infty} e^{-\theta\ell} = \frac{e^{-\theta}}{1 - e^{-\theta}},$$

which is a particular case of (5.9) with $c_{1,1} := 1$. Assume now that (5.9) is valid for some $q \geq 1$. Then, differentiating identities (5.8) and (5.9) with respect to θ , we obtain

$$\begin{aligned} S_{q+1}(\theta) &= -\frac{d}{d\theta} S_q(\theta) = \sum_{j=1}^q c_{j,q} \left(\frac{j e^{-\theta j}}{(1 - e^{-\theta})^j} + \frac{j e^{-\theta(j+1)}}{(1 - e^{-\theta})^{j+1}} \right) \\ &= \sum_{j=1}^{q+1} c_{j,q+1} \frac{e^{-\theta j}}{(1 - e^{-\theta})^j}, \end{aligned}$$

where we set

$$c_{j,q+1} := \begin{cases} c_{1,q}, & j = 1, \\ j c_{j,q} + (j-1) c_{j-1,q}, & 2 \leq j \leq q, \\ q c_{q,q}, & j = q+1. \end{cases}$$

In particular, $c_{q+1,q+1} = q c_{q,q} = q(q-1)! = q!$. Thus, formula (5.9) holds for $q+1$ and hence, by induction, for all $q \geq 1$. \square

Lemma 5.5. *For each $q \in \mathbb{N}$, there exists a positive constant C_q such that, for all $\theta > 0$,*

$$0 < S_q(\theta) \leq \frac{C_q e^{-\theta}}{(1 - e^{-\theta})^q}. \quad (5.10)$$

Proof. Observe that for $j = 1, \dots, q$ and all $\theta > 0$

$$\frac{e^{-\theta j}}{(1 - e^{-\theta})^j} \leq \frac{e^{-\theta}}{(1 - e^{-\theta})^q}.$$

Substituting these inequalities into (5.9) and recalling that the coefficients $c_{j,q}$ are positive, we obtain (5.10) with $C_q := \sum_{j=1}^q c_{j,q}$. \square

5.3. Asymptotics of moment sums

According to (2.17) and (3.2), the cumulants of $N_\lambda = \sum_{\ell=1}^{\infty} \ell \nu_\ell$ are given by

$$\kappa_q[N_\lambda] = \sum_{\ell=1}^{\infty} \ell^q \kappa_q(\ell) = \sum_{\ell=1}^{\infty} \ell^q \sum_{k=1}^{\infty} k^q a_k e^{-k\alpha_n \ell}, \quad q \in \mathbb{N}. \quad (5.11)$$

Lemma 5.6. *For each $q \in \mathbb{N}$,*

$$\kappa_q[N_\lambda] \asymp n^{(q+1)/2}, \quad n \rightarrow \infty. \quad (5.12)$$

Proof. Used notation (5.8), we can rewrite equation (5.11) as

$$\kappa_q[N_\lambda] = \sum_{k=1}^{\infty} k^q a_k S_{q+1}(k\alpha_n). \quad (5.13)$$

By Lemma 5.5 and inequality (3.5), each term in series (5.13) is bounded, uniformly in k , by

$$k^q |a_k| \frac{C_{q+1} e^{-k\alpha_n}}{(1 - e^{-k\alpha_n})^{q+1}} = \frac{O(1) |a_k|}{k\alpha_n^{q+1}}.$$

Hence, expanding $S_{q+1}(k\alpha_n)$ by Lemma 5.4, we can pass to the limit in (5.13) as $\alpha_n \rightarrow 0$,

$$\begin{aligned} \alpha_n^{q+1} \varkappa_q[N_\lambda] &= \sum_{j=1}^{q+1} c_{j,q+1} \sum_{k=1}^{\infty} k^q a_k \frac{\alpha_n^{q+1} e^{-kj\alpha_n}}{(1 - e^{-k\alpha_n})^j} \\ &\rightarrow c_{q+1,q+1} \sum_{k=1}^{\infty} \frac{a_k}{k} = q! A(1) = q! \kappa^2. \end{aligned} \quad (5.14)$$

Finally, according to (3.2) we have $\alpha_n^{q+1} \asymp n^{-(q+1)/2}$, and hence (5.14) implies (5.12). \square

In view of relation (2.18), Lemma 5.6 implies the following asymptotics.

Lemma 5.7. *For each $q \in \mathbb{N}$, we have $E_z(N_\lambda^q) \asymp n^{(q+1)/2}$ as $n \rightarrow \infty$.*

There is also a similar upper estimate for the centered absolute moments.

Lemma 5.8. *For each $q \in \mathbb{N}$,*

$$E_z |N_\lambda - E_z(N_\lambda)|^q = O(n^{(q+1)/2}), \quad n \rightarrow \infty.$$

Proof. Applying an inequality similar to (5.7), we obtain

$$E_z |N_\lambda - E_z(N_\lambda)|^q \leq 2^{q-1} E_z(N_\lambda^q) \asymp n^{(q+1)/2},$$

according to Lemma 5.7. \square

Since $\sum_{\ell=1}^{\infty} \ell^q \nu_\ell^q \leq N_\lambda^q$, Lemma 5.7 immediately yields one more corollary.

Lemma 5.9. *For each $q \in \mathbb{N}$,*

$$\sum_{\ell=1}^{\infty} \ell^q m_q(\ell) = O(|n|^{(q+1)/2}), \quad n \rightarrow \infty.$$

We shall also need a two-sided third-order estimate as follows.

Lemma 5.10. *As $n \rightarrow \infty$,*

$$\sum_{\ell=1}^{\infty} |\ell|^3 \mu_3(\ell) \asymp n^2,$$

where $\mu_3(\ell) = E_z |\nu_\ell^0|^3$ (see (5.6)).

Proof. An upper bound $O(n^2)$ follows from (5.7) and Lemma 5.9. On the other hand,

$$\mu_3(\ell) = E_z |\nu_\ell - m_1(\ell)|^3 \geq E_z (\nu_\ell - m_1(\ell))^3 = \varkappa_3(\ell),$$

using that the third-order centered moment coincides with the third-order cumulant. Hence, on account of formula (5.11),

$$\sum_{\ell=1}^{\infty} |\ell|^3 \mu_3(\ell) \geq \sum_{\ell=1}^{\infty} \ell_1^3 \varkappa_3(\ell) = \varkappa_3[N_\lambda] \asymp n^2,$$

according to Lemma 5.6 (with $q = 3$). \square

Let us introduce the *Lyapunov ratio*

$$L_z := \frac{1}{\sigma_z^3} \sum_{\ell=1}^{\infty} |\ell|^3 \mu_3(\ell). \quad (5.15)$$

The next asymptotic estimate is an immediate consequence of Lemmas 5.1 and 5.10.

Lemma 5.11. *As $n \rightarrow \infty$, we have $L_z \asymp n^{-1/4}$.*

Let us also consider the cumulants of $Y_\lambda(x) = \sum_{\ell > x} \nu_\ell$ given by (see (2.12) and (3.2))

$$\kappa_q[Y_\lambda(x)] = \sum_{\ell > x} \kappa_q(\ell) = \sum_{\ell > x} \sum_{k=1}^{\infty} k^q a_k e^{-k\alpha_n \ell}, \quad q \in \mathbb{N}.$$

Adapting the proof of Lemma 5.2 we easily obtain the following result.

Lemma 5.12. *For any $\delta > 0$ we have, uniformly in $x \geq \delta$,*

$$\kappa_q[Y_\lambda(xn^{1/2})] = O(n^{1/2}), \quad n \rightarrow \infty.$$

Likewise Lemmas 5.7 and 5.8, the result of Lemma 5.12 implies a similar asymptotic bound on the centered absolute moments of $Y_\lambda(\cdot)$.

Lemma 5.13. *For each $q \in \mathbb{N}$ and any $\delta > 0$ we have, uniformly in $x \geq \delta$,*

$$E_z |Y_\lambda(xn^{1/2}) - E_z[Y_\lambda(xn^{1/2})]| = O(n^{1/2}), \quad n \rightarrow \infty.$$

6. Local limit theorem

The role of a local limit theorem in our approach is to yield the asymptotics of the probability $Q_z\{N_\lambda = n\} \equiv Q_z(\Pi_n)$ appearing in the representation of the measure P_n as a conditional distribution, $P_n(\cdot) = Q_z(\cdot | \Pi_n) = Q_z(\cdot)/Q_z(\Pi_n)$.

6.1. Statement of the theorem

As before, denote $a_z := E_z(N_\lambda)$, $\sigma_z^2 := \text{Var}_z(N_\lambda) = E_z(N_\lambda - a_z)^2$. Consider the probability density of a normal distribution $\mathcal{N}(a_z, \sigma_z^2)$ (with mean a_z and variance σ_z^2), given by

$$\phi_{a_z, \sigma_z}(x) = \frac{1}{\sqrt{2\pi} \sigma_z} \exp\left(-\frac{1}{2}(x - a_z)^2/\sigma_z^2\right), \quad x \in \mathbb{R}. \quad (6.1)$$

Theorem 6.1. *Let $A^+(1) < \infty$ and Assumption 2.1 hold. Then, uniformly in $m \in \mathbb{Z}_+$,*

$$Q_z\{N_\lambda = m\} = \phi_{a_z, \sigma_z}(m) + O(n^{-1}), \quad n \rightarrow \infty. \quad (6.2)$$

Corollary 6.2. *Further to the conditions of Theorem 6.1, assume that $A^+(\eta) < \infty$ for some $\eta \in (0, \frac{1}{2})$. Then*

$$Q_z\{N_\lambda = n\} \sim \frac{\kappa^{3/2}}{2\sqrt{\pi A(1)} n^{3/4}}, \quad n \rightarrow \infty. \quad (6.3)$$

Remark 6.1. The cumulant asymptotics (Lemma 5.6), together with the asymptotics of the first two moments (Theorem 3.1 and Lemma 5.1), immediately lead to a central limit theorem for the random variable $N_\lambda = \sum_{\ell=1}^{\infty} \ell \nu_\ell$ (which is of course consistent with Theorem 6.1).

Theorem 6.3 (CLT). *The distribution of the random variable $\sigma_z^{-1}(N_\lambda - a_z)$ converges weakly, as $n \rightarrow \infty$, to the standard normal distribution $\mathcal{N}(0, 1)$.*

Before proving Theorem 6.1, we have to make some technical preparations.

6.2. Estimates of the characteristic functions

Recall from Section 2.1 that, with respect to the measure Q_z , the random variables $\{\nu_x\}$ are independent and have characteristic functions (2.9). Hence, the characteristic function $\varphi_{N_\lambda}(t) := E_z(e^{itN_\lambda})$ of the sum $N_\lambda = \sum_{\ell=1}^{\infty} \ell \nu_\ell$ is given by

$$\varphi_{N_\lambda}(t) = \prod_{\ell=1}^{\infty} \varphi_\nu(t\ell; \ell) = \prod_{\ell=1}^{\infty} \frac{f(z^\ell e^{it\ell})}{f(z^\ell)}, \quad t \in \mathbb{R}. \quad (6.4)$$

Let us start with a general absolute estimate for the characteristic function of a centered random variable (for a proof, see [3, Lemma 7.10]).

Lemma 6.4. *Let $\varphi_{\nu^0}(t; \ell) := E_z(e^{it\nu_\ell^0})$ be the characteristic function of the random variable $\nu_\ell^0 = \nu_\ell - E_z(\nu_\ell)$. Then*

$$|\varphi_{\nu^0}(t; \ell)| \leq \exp\left\{-\frac{1}{2}\mu_2(\ell)t^2 + \frac{1}{3}\mu_3(\ell)|t|^3\right\}, \quad t \in \mathbb{R}, \quad (6.5)$$

where $\mu_q(\ell) := E_z|\nu_\ell^0|^q$.

The next lemma provides two estimates (proved in [3, Lemmas 7.11 and 7.12]) for the characteristic function $\varphi_{N_\lambda^0}(t) := E_z(e^{itN_\lambda^0})$ of the centered random variable $N_\lambda^0 = N_\lambda - a_z = \sum_{\ell=1}^{\infty} \ell \nu_\ell^0$. Recall that the Lyapunov ratio L_z is defined in (5.15).

Lemma 6.5. (a) *For all $t \in \mathbb{R}$,*

$$|\varphi_{N_\lambda^0}(t\sigma_z^{-1})| \leq \exp\left\{-\frac{1}{2}|t|^2 + \frac{1}{3}L_z|t|^3\right\}. \quad (6.6)$$

(b) *If $|t| \leq L_z^{-1}$ then*

$$\left|\varphi_{N_\lambda^0}(t\sigma_z^{-1}) - e^{-|t|^2/2}\right| \leq 16L_z|t|^3 e^{-|t|^2/6}. \quad (6.7)$$

Let us also prove the following global bound (cf. [3, Lemma 7.13]).

Lemma 6.6. *As in Theorem 6.1, suppose that Assumption 2.1 is satisfied. Then*

$$|\varphi_{N_\lambda^0}(t)| \leq e^{-C_0 J_n(t)}, \quad t \in \mathbb{R}, \quad (6.8)$$

where C_0 is a positive constant and

$$J_n(t) := \sum_{\ell=1}^{\infty} e^{-\alpha_n \ell} (1 - \cos t\ell) \geq 0. \quad (6.9)$$

Proof. From (6.4) we have

$$|\varphi_{N_\lambda^0}(t)| = |\varphi_{N_\lambda}(t)| = \exp \left\{ \sum_{\ell=1}^{\infty} \ln |\varphi_\nu(t\ell; \ell)| \right\}. \quad (6.10)$$

Recall that under Assumption 2.1 we have, according to (2.21) and (2.22),

$$\ln |\varphi_\nu(t\ell; \ell)| = \frac{1}{2} \ln \frac{f(z^\ell e^{it\ell}) f(z^\ell e^{-it\ell})}{f(z^\ell)^2} \leq -\delta_1 c_1 z^\ell (1 - \cos t\ell),$$

with $\delta_1 > 0$ and $c_1 > 0$. Utilizing this estimate under the sum in (6.10) and recalling notation (3.2), we arrive at (6.8) with $C_0 := \delta_1 c_1 > 0$. \square

6.3. Proof of Theorem 6.1 and Corollary 6.2

Let us first deduce the corollary from the theorem.

Proof of Corollary 6.2. According to Theorem 4.1, $a_z = n + O(n^{1/2+\eta/2})$ with $\eta \in (0, \frac{1}{2})$. Together with Lemma 5.1, this implies $(n - a_z)/\sigma_z = O(n^{-1/4+\eta/2}) = o(1)$. Hence,

$$\phi_{a_z, \sigma_z}(n) = \frac{1}{\sqrt{2\pi} \sigma_z} \exp \left\{ -\frac{1}{2} (n - a_z)^2 \sigma_z^{-2} \right\} \sim \frac{1}{\sqrt{2\pi}} \cdot \frac{\kappa^{3/2}}{\sqrt{2A(1)} n^{3/4}},$$

and (6.3) now readily follows from (6.2). \square

Proof of Theorem 6.1. By definition, the characteristic function of the random variable $N_\lambda^0 = N_\lambda - a_z$ is given by the Fourier series

$$\varphi_{N_\lambda^0}(t) = \sum_{m=0}^{\infty} Q_z \{N_\lambda = m\} e^{it(m-a_z)}, \quad t \in \mathbb{R},$$

hence the Fourier coefficients are expressed as

$$Q_z \{N_\lambda = m\} = \frac{1}{2\pi} \int_T e^{it(m-a_z)} \varphi_{N_\lambda^0}(t) dt, \quad m \in \mathbb{Z}_+, \quad (6.11)$$

where $T := [-\pi, \pi]$. On the other hand, the characteristic function corresponding to the normal probability density $\phi_{a_z, \sigma_z}(x)$ (see (6.1)) is given by

$$\varphi_{a_z, \sigma_z}(t) = e^{ita_z - t^2 \sigma_z^2 / 2}, \quad t \in \mathbb{R},$$

so by the Fourier inversion formula

$$\phi_{a_z, \sigma_z}(m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it(m-a_z) - t^2 \sigma_z^2 / 2} dt, \quad m \in \mathbb{Z}_+. \quad (6.12)$$

Denote $D_z := \{t \in \mathbb{R} : |t| > (L_z \sigma_z)^{-1}\}$ and observe that if $t \notin D_z$ then, according to Lemmas 5.1 and 5.11, $|t| \leq L_z^{-1} \sigma_z^{-1} = O(n^{-1/2}) = o(1)$, which implies that $t \in T$. Using this observation and subtracting (6.12) from (6.11), we get, uniformly in $m \in \mathbb{Z}_+$,

$$|Q_z \{N_\lambda = m\} - \phi_{a_z, \sigma_z}(m)| \leq I_1 + I_2 + I_3, \quad (6.13)$$

where

$$I_1 := \frac{1}{2\pi} \int_{D_z^c} |\varphi_{N_\lambda^0}(t) - e^{-t^2\sigma_z^2/2}| dt, \quad I_2 := \frac{1}{2\pi} \int_{D_z} e^{-t^2\sigma_z^2/2} dt,$$

$$I_3 := \frac{1}{2\pi} \int_{T \cap D_z} |\varphi_{N_\lambda^0}(t)| dt.$$

By the substitution $t = y\sigma_z^{-1}$, the integral I_1 is reduced to

$$I_1 = \frac{1}{2\pi\sigma_z} \int_{|y| \leq L_z^{-1}} |\varphi_{N_\lambda^0}(y\sigma_z^{-1}) - e^{-y^2/2}| dy$$

$$= O(1)\sigma_z^{-1}L_z \int_{-\infty}^{\infty} |y|^3 e^{-y^2/6} dy = O(n^{-1}), \quad (6.14)$$

on account of Lemmas 5.1, 5.11 and 6.5(b). Similarly, passing to the polar coordinates we get, again using Lemmas 5.1 and 5.11,

$$I_2 = \frac{1}{\pi\sigma_z} \int_{L_z^{-1}}^{\infty} r e^{-r^2/2} dr = O(n^{-3/4}) e^{-L_z^{-2}/2} = o(n^{-1}). \quad (6.15)$$

Finally, let us turn to I_3 . Using Lemma 6.6, we obtain

$$I_3 = O(1) \int_{T \cap D_z} e^{-C_0 J_n(t)} dt, \quad (6.16)$$

where $J_n(t)$ is given by (6.9). If $t \in D_z$ then $|t| > \delta\alpha_n$ for a suitable (small enough) constant $\delta > 0$, for otherwise from (3.2) and Lemmas 5.1 and 5.11 it would follow

$$1 < L_z\sigma_z|t| \leq L_z\sigma_z\delta\alpha_n = O(\delta) \rightarrow 0 \quad \text{as } \delta \rightarrow 0+,$$

which is a contradiction. Hence, estimate (6.16) is reduced to

$$I_3 = O(1) \int_{|t| > \delta\alpha_n} e^{-C_0 J_n(t)} dt. \quad (6.17)$$

Furthermore, from (6.9) we obtain

$$J_n(t) = \sum_{\ell=1}^{\infty} e^{-\alpha_n \ell} (1 - \Re e^{it\ell}) = \frac{1}{1 - e^{-\alpha_n}} - \Re \left(\frac{e^{it}}{1 - e^{-\alpha_n + i\lambda_n}} \right)$$

$$\geq \frac{1}{1 - e^{-\alpha_n}} - \frac{1}{|1 - e^{-\alpha_n + i\lambda}|}, \quad (6.18)$$

because $\Re u \leq |u|$ for any $u \in \mathbb{C}$. Since $\delta\alpha_n \leq |t| \leq \pi$, we have

$$|1 - e^{-\alpha_n + it}| \geq |1 - e^{-\alpha_n + i\delta\alpha_n}| \sim \alpha_n(1 + \delta^2)^{1/2} \quad (\alpha_n \rightarrow 0).$$

Substituting this estimate into (6.18), we conclude that $J_n(t)$ is asymptotically bounded from below by $C(\delta)\alpha_n^{-1} \asymp n^{1/2}$ (with some constant $C(\delta) > 0$), uniformly in t such that $\delta\alpha_n \leq |t| \leq \pi$. Thus, the integral in (6.17) is bounded by $O(1) \exp(-\text{const} \cdot n^{1/2}) = o(n^{-1})$.

Hence, recalling estimates (6.14) and (6.15), we see that the right-hand side of inequality (6.13) admits an asymptotic bound $O(n^{-1})$, which completes the proof of the theorem. \square

7. Limit shape

7.1. Proof of the limit shape results

In this section, we suppose that Assumptions 2.1 and 3.1 are fulfilled. Let us first establish the limit shape under the measure Q_z . Recall that $\omega^*(x) = \kappa^{-1}H(e^{-\kappa x})$ (see (1.7)), where $H(s) = \ln f(s) = \sum_{k=1}^{\infty} a_k s^k$ and $\kappa^2 = \sum_{k=1}^{\infty} a_k/k$ (see (2.6) and (3.4)).

Theorem 7.1. *Assume that $A^+(1) < \infty$. Then, for each $\delta > 0$ and any $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} Q_z \left\{ \lambda \in \Pi : \sup_{x \geq \delta} |n^{-1/2} Y_\lambda(xn^{1/2}) - \omega^*(x)| \leq \varepsilon \right\} = 1.$$

Proof. By Theorem 3.3, the expectation of $n^{-1/2} Y_\lambda(xn^{1/2})$ converges, uniformly in $x \geq \delta$, to $\omega^*(x)$ as $n \rightarrow \infty$. Therefore, we only need to check that, for each $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} Q_z \left\{ \lambda \in \Pi : \sup_{x \geq \delta} n^{-1/2} |Y_\lambda(xn^{1/2}) - E_z[Y_\lambda(xn^{1/2})]| > \varepsilon \right\} = 0.$$

From the definition of the random process $Y_\lambda(\cdot)$ (see (1.1)), for any $0 \leq s < t$ we have $Y_\lambda(t) - Y_\lambda(s) = \sum_{s < \ell \leq t} \nu_\ell$, and it follows that $Y_\lambda(t)$ has independent increments. Hence, $Y_\lambda^0(t) := Y_\lambda(t) - E_z[Y_\lambda(t)]$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{\nu_\ell, \ell \leq t\}$. From (1.1) it is also clear that $Y_\lambda^0(t)$ is càdlàg (i.e., its paths are everywhere right-continuous and have left limits). Therefore, applying the Kolmogorov–Doob submartingale inequality (see, e.g., [29, Corollary 2.1, p. 14]) and using Lemma 5.2, we obtain

$$Q_z \left\{ \sup_{x \geq \delta} |Y_\lambda^0(xn^{1/2})| > \varepsilon n^{1/2} \right\} \leq \frac{\sup_{x \geq \delta} \text{Var}_z[Y_\lambda(xn^{1/2})]}{\varepsilon^2 n} = O(n^{-1/2}) \rightarrow 0,$$

and the theorem is proved. \square

We are finally ready to prove our main result about the limit shape under the measure P_n (cf. Theorem 1.1).

Theorem 7.2. *Let $A^+(\eta) < \infty$ with some $\eta \in (0, \frac{1}{2})$. Then, for each $\delta > 0$ and any $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} P_n \left\{ \lambda \in \Pi_n : \sup_{x \geq \delta} |n^{-1/2} Y_\lambda(xn^{1/2}) - \omega^*(x)| \leq \varepsilon \right\} = 1.$$

Proof. Like in the proof of Theorem 7.1, the claim is reduced to the limit

$$\lim_{n \rightarrow \infty} P_n \left\{ \lambda \in \Pi_n : \sup_{x \geq \delta} |Y_\lambda^0(xn^{1/2})| > \varepsilon n^{1/2} \right\} = 0. \quad (7.1)$$

Using (2.5) we get the estimate

$$P_n \left\{ \sup_{x \geq \delta} |Y_\lambda^0(xn^{1/2})| > \varepsilon n^{1/2} \right\} \leq \frac{Q_z \left\{ \sup_{x \geq \delta} |Y_\lambda^0(xn^{1/2})| > \varepsilon n^{1/2} \right\}}{Q_z \{N_\lambda = n\}}. \quad (7.2)$$

Furthermore, similarly as in the proof of Theorem 7.1, by the Kolmogorov–Doob submartingale inequality and Lemma 5.13 (with $q = 3$) we have

$$Q_z \left\{ \sup_{x \geq \delta} |Y_\lambda^0(xn^{1/2})| > \varepsilon n^{1/2} \right\} \leq \frac{\sup_{t \geq \delta} E_z |Y_\lambda(xn^{1/2}) - E_z(Y_\lambda(xn^{1/2}))|^3}{\varepsilon^3 n^{3/2}} = O(n^{-1}).$$

On the other hand, $Q_z \{N_\lambda = n\} \asymp n^{-3/4}$ by Corollary 6.2. As a result, the right-hand side of (7.2) is dominated by $O(n^{-1/4}) \rightarrow 0$, and so the limit in (7.1) follows. \square

7.2. Examples (continued from Section 2.3)

As was mentioned in Section 2.3, the partition ensembles described in Examples 2.1–2.4 satisfy all the conditions needed for Theorem 7.2, except for the special case $\rho = 1$ in Example 2.3, where the assumption $A^+(\eta) < \infty$ (with $0 < \eta < \frac{1}{2}$) breaks down.

Let us list the explicit equations for the limit shape ω^* in all the examples. Starting with Example 2.1, we have $H(s) = -r \ln(1 - \rho s)$ and, according to (2.27) and (3.4),

$$\kappa^2 = r \sum_{k=1}^{\infty} \frac{\rho^k}{k^2} = \text{Li}_2(\rho),$$

where $\text{Li}_2(\cdot)$ is the dilogarithm. The limit shape ω^* is therefore given by the equation

$$y = -\frac{r}{\kappa} \ln(1 - \rho e^{-\kappa x}), \quad x \geq 0.$$

In particular, if $r = 1$ and $\rho = 1$, then $\kappa^2 = \text{Li}_2(1) = \pi^2/6$ and we recover equation (1.3).

In Example 2.2, we have $H(s) = r \ln(1 + \rho s)$ and, according to (2.31) and (3.4),

$$\kappa^2 = r \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \rho^k}{k^2} = -\text{Li}_2(-\rho) \equiv \text{Li}_2(\rho) - \frac{1}{2} \text{Li}_2(\rho^2),$$

while the limit shape ω^* is given by the equation

$$y = \frac{r}{\kappa} \ln(1 + \rho e^{-\kappa x}), \quad x \geq 0.$$

In particular, if $r = 1$ and $\rho = 1$, then $\kappa^2 = \frac{1}{2} \text{Li}_2(1) = \pi^2/12$, and we recover equation (1.4).

In Example 2.3, we have $H(s) = rs/(1 - \rho s)$ and, according to (2.35) and (3.4),

$$\kappa^2 = r \sum_{k=1}^{\infty} \frac{\rho^{k-1}}{k} = -\frac{r \ln(1 - \rho)}{\rho} < \infty$$

for all $\rho \in [0, 1)$. Hence, the limit shape ω^* is given by the equation

$$y = \frac{r e^{-\kappa x}}{\kappa(1 - e^{-\kappa x})}, \quad x \geq 0.$$

The case $\rho = 1$ is not covered by our results; to get a sensible limit shape, one can modify formula (2.33) by taking a truncated sum under the exponent, leading to

$$H(s) = r \sum_{k=1}^M s^k = \frac{rs(1 - s^M)}{1 - s}$$

and $\kappa = r \sum_{k=1}^M k^{-1}$, with the corresponding limit shape easily specified from equation (1.7).

Finally, let us consider Example 2.4. From (2.36) we have

$$H(s) = r \ln \left(\frac{-\ln(1 - \rho s)}{\rho s} \right), \quad (7.3)$$

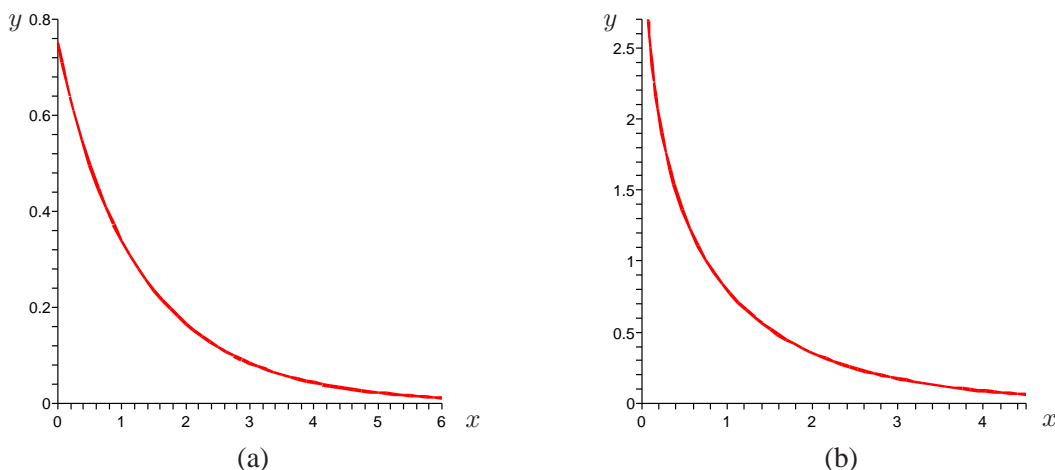


Figure 3: The limit shape ω^* in Example 2.4, with the function $H(s)$ given by equation (7.3): (a) $r = 1.5$, $\rho = 0.5$; (b) $r = 1.5$, $\rho = 1$.

and it is hard to get the power series expansion explicitly, even with a computer. For numerical calculations, it is more convenient to use an alternative formula for κ provided by Lemma 3.2,

$$\kappa^2 = \int_0^1 s^{-1} H(s) \, ds.$$

For an illustrative example, taking $r = 1.5$ and $\rho = 0.5$ we computed $\kappa = 0.6518118431$, and the limit shape can now be plotted from the explicit equation (1.7) with the function $H(e^{-\kappa x})$ evaluated from formula (7.3) (see Fig. 3a). For a comparison, we also plotted the limit shape with parameters $r = 1.5$ and $\rho = 1$, giving $\kappa = 1.045485952$; the corresponding plot of the limit shape is shown in Fig. 3b.

7.3. Concluding remarks

It seems natural to try and relax the simplifying conditions of Assumption 2.1 that facilitated the proof of the local limit theorem (Theorem 6.1). We also hope that it may be possible to combine the methods of the present work with the techniques developed in [28] and to improve the limit shape analysis in the general, non-conservative case (i.e., with nonconstant sequences of powers b_ℓ for $f_\ell(s) = f(s)^{b_\ell}$, cf. (1.5)). We will address these issues elsewhere.

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